Proof and problem posing about periodic functions

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Introduction

This article presents two classroom episodes in which students were exposed to the value of asking questions and to the different roles played by proof in mathematics. Having students ask questions is a way to address what de Villiers labels a 'distorted perspective of mathematical creativity as being always purely deductive' (de Villiers, 1997, p.15). He claims that 'the false impression is sometimes created that mathematicians are only problems solvers who spend most of their time trying to solve already given problems' (ibid).

The conversation in the two episodes is outlined in the paper. The setting was a classroom of fifteen good high-school students, who were studying calculus. These episodes occurred spontaneously, after the discussion of certain theorems during the lessons. Each one of these theorems will be labelled trigger of the episode, since they inspired students to ask the initial question that led to the whole episode. All over her work, it appears that their teacher approaches mathematics teaching following Marcel Proust's (Ref. 1) saying, 'The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.'

The content she teaches appears in the curriculum and is the same content all her colleagues teach, so the 'landscape' is not new. However, it seems she teaches her students to develop 'new eyes' and to look at the same landscape all the other students see, with their new eyes.

The teacher produced the documentation of the discussions immediately after the end of the lesson, based on her memory and the students' notes. She was encouraged to do so by the writer who was witness of her enthusiasm in the teacher's staff room.

Episode 1

Trigger
The derivative of a periodic function is also a periodic function.

Student question
Is every antiderivative of a periodic function also a periodic function?

Construction of an example by a student
Consider the periodic function $h(x) = cos(x)$. Is it possible that h is the derivative of a certain non-periodic function f? If $f(x) = sin(x)$, then $f'(x) = h(x)$ but in this case f is periodic. Looking at a sketch of the function $y = sin(x)$, he suggested considering it by segments and moving up or down each part of the graph (see Figure 1).

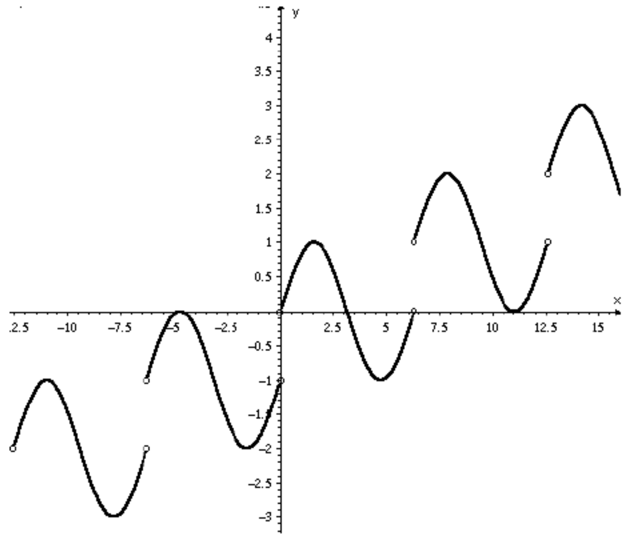

Figure 1. Graph of a non-periodic function, whose derivative is a periodic function.

Then, the student wrote on the board the non-periodic function that he had previously described by means of its graph:

$$f(x) = \begin{cases} \vdots \\ \sin(x)+1 & 2\delta \le x < 4\delta \\ \sin(x) & 0 \le x < 2\delta \\ \sin(x)-1 & -2\delta \le x < 0 \\ \vdots \end{cases}$$

He concluded that an antiderivative of a periodic function is not necessarily periodic since the function f is not periodic but f' is a periodic function since '$f'(x) = cos(x)$ for every real x such that $x \neq 2pk$ (k any integer number)'.

Student question
We know how to prove that a function is periodic, but how do you prove that a function is not periodic? How do we prove that f is indeed not periodic? You can see that from its graph, but how do you explain why it is so?

In other words, the student who asked the question believed that the property 'an antiderivative of a periodic function is not necessarily periodic' was true but asked for a proof in order to get an insight into why it is true. Following de Villiers' model, in this case proof did not play the role of *verification* nor the role of *conviction*, but the role of *explanation* (de Villiers, 1999).

One of the students (S1) tried to explain that f cannot be periodic because it has only one real root. One of the other students (S2) did not see the connection between these facts, and thought he could build a periodic function with any number of roots. He presented the sketch in Figure 2 to exemplify a 'periodic function with exactly three roots'.

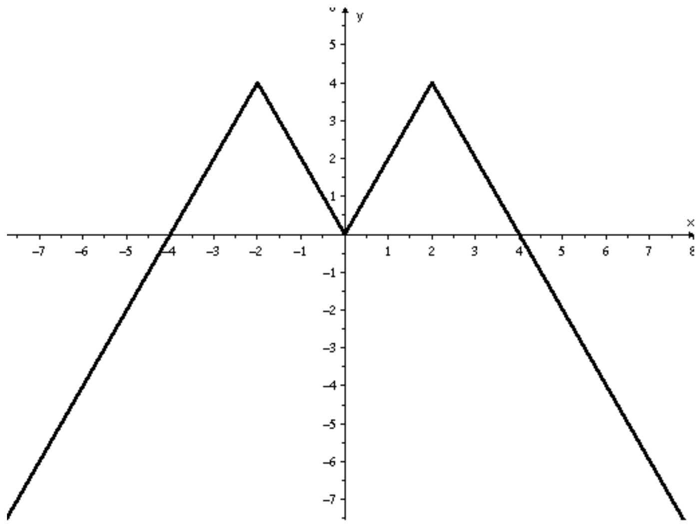

Figure 2. Graph of Daniel's function: a 'periodic' function with exactly three roots.

Most of the students agreed that this function had three roots but the problem was to decide whether it was periodic. S1 tried to convince the others that it was not periodic by looking at the points of intersection of its graph and the line $y = 4$. He reasoned as follows: if this function were periodic with period p, the equation $f(x) = 4$ must have had an infinite number of solutions including all the numbers of the form $2 + n{\cdot}p$ for every integer n. But it can be easily proved using algebraic tools that this equation has exactly two roots. This fact contradicts the existence of a period p.

The teacher rescued a statement formulated by S2 some minutes before and wrote it on the board: 'There are periodic functions with any number of roots'.

Then she asked the students for reactions. One of them (S3) asked whether is it possible at all to have a periodic function with a finite number of roots. S1 remarked that a periodic function may not have a root at all and when he was challenged by the teacher to present an example of such a function, he chose the function $y = \sin(x) + 10$ that indeed has no roots since for every real number x, $11 \geq \sin(x) + 10 \geq 9$. From here the students concluded that a periodic function may have no roots at all, or it may have an infinite number of roots.

In this situation, S2 tried to build an argument in order to refute what he

believed was a false statement. He was aware of the fact that one single counter-example suffices to contradict a given conjecture, but he generated a 'counter-example' that turned to be a non-example, since the function he presented was not periodic. This case shows some of the difficulties students can have building appropriate counter-examples to refute a statement, which is a problem that Zaslavsky and Ron (1998) comment on.

It is important to notice that although the teacher identified the potential embedded in S2's statement ('There are periodic functions with any number of roots'), she did not interrupt the dynamics of the lesson. She considered it worthwhile discussing it, so she presented it again. Her attitude reminds us of one of George Polya's *Ten Commandments for Teachers*: 'Do not give away your whole secret at once — let the students guess before you tell it — let them find out by themselves as much as is feasible' (Polya, 1981, p. 116). The journey in which the students got involved during this episode appears much more fruitful and instructive than its last station, the conclusion itself.

Episode 2

Trigger

The product of two even functions is an even function and the product of two odd functions is also an even function

Student question

Is the product of two periodic functions also a periodic function?

Analysis of an example by a student

Consider the product of $f(x) = \sin(x)$ and $g(x) = \cos(x)$.

Their product is $h(x) = \dfrac{\sin(2x)}{2}$, which is periodic.

Another student argued that while the principal period of the functions f and g is 2π, the principal period of their product is π. So, their product is indeed a periodic function but its principal period is not the same.

Student question

The example shown constitutes a special case of a pair of functions with the same principal period. Is the product of any two functions with the same period also a periodic function?

The teacher asked the student to look for the answer. He came to the board and used an algebraic approach, using the definition of periodic function:

$$\left.\begin{array}{l} \exists p \mid \forall x \in D_f \;\; f(x+p) = f(x) \\ \exists p \mid \forall x \in D_g \;\; g(x+p) = g(x) \end{array}\right\} \Rightarrow \exists p \mid \forall x \in D_f \cap D_g \;\; f(x+p) \,.\, g(x+p) = f(x) \,.\, g(x)$$

From here, he was able to conclude that the function $f.g$ is indeed a periodic function. His approach allowed him to answer his own question. He really did not know the answer in advance so, in this case, he approached proving not as a verification task but as a discovery tool (de Villiers, 1999).

Student question
The product of two periodic functions with the same principal period is also a periodic function. Is its period always smaller that theirs?

This question was not answered. Instead a new question was formulated, which fitted the 'What if Not' strategy to problem posing (Brown & Walters, 1983). This strategy to generate new problems is based on changing the conditions of a current problem. For example, given a mathematics theorem, the student may be asked to identify its attributes. After a discussion of these attributes, the student may ask, 'What if some or all of the given attributes are not true?' Through this discussion, the students generate new problems.

Student question
Is the product of any two periodic functions with different principal periods also a periodic function?

Analysis of an example by a student (S1)
Consider the product of $f(x) = \cos(x)$ and $g(x) = \tan(x)$. Their product is $h(x) = \sin(x)$ (for every $x \neq \pi k$, $k \in Z$). While the principal periods of f and g are 2π and π respectively, the principal period of h is 2π.

Formulation of a conjecture (S2)
The product of two periodic functions f and g with principal periods p_1 and p_2 respectively is a periodic function and its principal period is the maximum of $\{p_1, p_2\}$.

Analysis of an example (S3)
Consider the product of
$f(x) = \sin\left(\frac{x}{2}\right)$ and $g(x) = \sin\left(\frac{x}{3}\right)$
whose principal periods are 4π and 6π respectively.
Test whether the period of the function
$h(x) = \sin\left(\frac{x}{2}\right) \cdot \sin\left(\frac{x}{3}\right)$ is 6π.

Refutation of the conjecture (S3 at S1's suggestion and S4). Taking $x = \frac{\pi}{2}$:

$$h\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{4} \text{ and}$$

$$h\left(\frac{\pi}{2} + 6\pi\right) = \sin\left(\frac{\pi}{4} + 3\pi\right) \cdot \sin\left(\frac{\pi}{6} + 2\pi\right) = (-1)\sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{6}\right) = -h\left(\frac{\pi}{2}\right)$$

Looking for the period of the function h (S5):

$$h\left(\frac{\pi}{2} + p\right) = \sin\left(\frac{\pi}{4} + \frac{p}{2}\right) \cdot \sin\left(\frac{\pi}{6} + \frac{p}{3}\right) = \sin\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{\pi}{6}\right) = h\left(\frac{\pi}{2}\right)$$

Since the numbers $\frac{p}{2}, \frac{p}{3}$ must be multiples of 2π, p must be 12π.

Formulation of a conjecture (S1) — Generalisation of the former result:
If p_1 and p_2 are the principal periods of two functions f and g respectively, their product is a periodic function and its principal period is 'something like the least common multiple of p_1 and p_2'.
Verification of the conjecture with a new example (S1):
The principal period of the function

$$h(x) = \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{5}\right)$$

is 20π since the principal period of

$$f(x) = \sin\left(\frac{x}{2}\right)$$

is 4π, the principal period of

$$g(x) = \cos\left(\frac{x}{5}\right)$$

is 10π and the lowest common multiple of 4π and 10π is 20π.

Student question (S4):
Does any couple of positive numbers p_1 and p_2 have a positive number q such that $\frac{q}{p_1}$ and $\frac{q}{p_2}$ are natural numbers?

Reformulation of the question (S5):
Does any couple of periodic functions f and g with principal periods p_1 and p_2 respectively have a positive number q that is the period of the function $f.g$?

Negative answer to the question (S6):
If a function has period $p_1 = 2p$ and the other function has period $p_2 = 1$, their product is a non-periodic function since there is no common multiple for the numbers $2p$ and 1.

Conclusion (S4):
$f(x) = \sin(x) \cdot \sin(2\pi x)$ is not a periodic function (see Figure 3). The product of two periodic functions is not necessarily a periodic function.

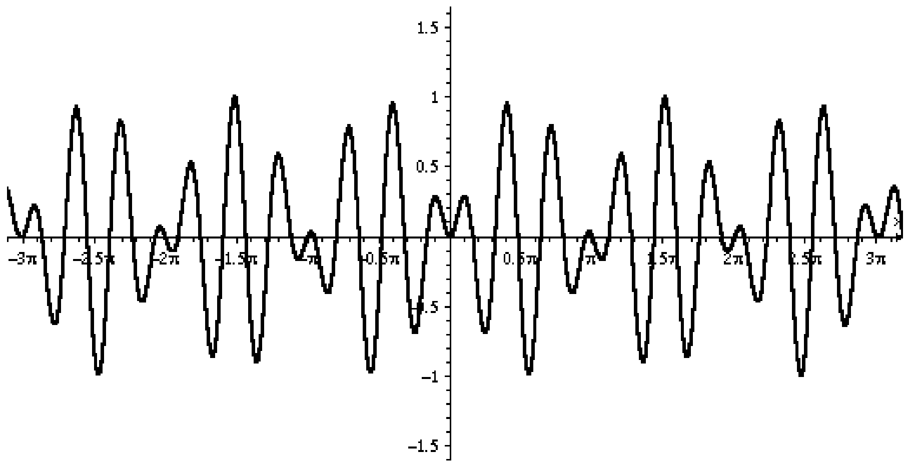

Figure 3: Graph of the non-periodic function $f(x) = \sin(x) \cdot \sin(2\pi x)$

Concluding remarks

The students involved in the episodes described in this article may be considered experts in Polya's terms:

> Quite often, when an idea that could be helpful presents itself, we do not appreciate it, for it is so inconspicuous. The expert has, perhaps, no more ideas than the inexperienced, but appreciates more what he has and uses it better. (Ref. 2)

The students were taught how to focus their attention on knowns, unknowns and restrictions of the problems they solved and they felt comfortable playing with them. In this way some of them became 'experts' in problem posing too. Moreover, different facets of the process of solving a mathematical problem emerged during these episodes:

a) representation of problem situations in a variety of forms (graphic, symbolic, verbal);
b) generalisation from an observation made in a particular case;
c) test of a conjecture;
d) distinction between valid and invalid arguments (e.g. reasoning from a counter-example, reasoning from an example);
e) explanation of how a certain conclusion was derived;
f) formulation of new questions.

As these episodes show, a classroom climate in which students formulate questions and not simply solve already given problems is an environment that may lead students to the appreciation of the nature of mathematical proofs as well as the different roles they play during a mathematical activity.

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Ref. (1) http://www.quoteworld.org/author.php?thetext=Marcel%20Proust&page=2

Ref. (2) http://www.mathacademy.com/pr/quotes/index.asp?ACTION=AUT&VAL=Polya